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Relativistic Kepler map

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Abstract

The relativistic generalization of the Kepler map describing diffusive excitation of the relativistic hydrogen-like atom in a monochromatic field is derived. It is shown that the trajectories which are regular in the non-relativistic case may become chaotic in the relativistic case under the same conditions.

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Microwave excitation and ionization of highly excited hydrogen atoms has been the subject of extensive theoretical (Jensen 1984, Casati *et al* 1987a, b, 1988, Delone *et al* 1983) as well as experimental (Bayfield and Koch 1974, Jensen *et al* 1991, Koch and Van Leeuwen 1998) investigation for the last three decades. A most interesting phenomenon in such an interaction is the chaotization of motion of the Kepler electron under the influence of the monochromatic field. Since 1974 when early experiments on chaotic ionization were performed (Bayfield and Koch 1974) great progress has been made in the theoretical as well as the experimental study of microwave ionization of the hydrogen atom. The first theoretical explanation of microwave ionization was given by Delone *et al* (1978) who assumed that it is of diffusive character. Resonance analysis based on the Chirikov criterion has shown itself to be a powerful tool, allowing analytical estimation of the critical value of the external field at which ionization will occur (Jensen 1984, Delone *et al* 1983).

Casati *et al* (1987a, b, 1988) introduced the so-called Kepler map. Approximating the effect at perturbation by an instantaneous impulse that is applied once per orbital period of motion, they replaced the Hamiltonian equation of motion with a mapping equation of motion which they called the Kepler map. Such a map greatly facilitates numerical investigation of the dynamics ionization process and even allows an analytical estimation of the threshold field strengths for the onset of chaos, the diffusion coefficient of the electron in energy space and other characteristics of the system (Casati *et al* 1987a, b, 1988, Kaulakis and Vilutis 1999). Another important advantage of the Kepler map is the fact that it can be locally approximated by the standard map (Casati *et al* 1988, Jensen *et al* 1981) which is a well studied model. Moreover, quantization of this map allows one to treat the quantum effects in the chaotic ionization of the hydrogen atom; by reducing its quantum dynamics to the quantum kicked rotator model, one can observe the quantum localization phenomenon, which arises from the fact that chaos is suppressed in the quantum chaos.

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In the present paper we will derive the relativistic generalization of the Kepler map which describes diffusive excitation and ionization of the relativistic hydrogen-like atom in a monochromatic field. We will derive the Kepler map for the one-dimensional relativistic hydrogen-like atom in a monochromatic field. By a relativistic atom one means the atom, of which the electron has a relativistic velocity due to the high charge of the atomic nucleus. Fast growing interest in the physics and chemistry of the actinides and transactinides is stimulating extensive study of heavy and super-heavy relativistic atoms. The study and synthesis of super-heavy elements is becoming one of the current problems of modern physics (Pershina and Fricke 1998, Holman 1999).

One of the main differences between relativistic and non-relativistic atoms, which leads to the additional difficulty in the theoretical study of relativistic atoms, is the fact that the motion of the atomic electrons in the relativistic case is described by equations of motion which are more nonlinear than in the non-relativistic case. Separate from super-heavy atoms, such a one-dimensional relativistic hydrogen-like atom is a convenient model for the study of dynamical chaos in relativistic systems. Note that up to now investigation of chaos in dynamical systems has mainly been limited by non-relativistic systems. However, the relativistic systems could be more interesting for classical quantum chaology, since they are more nonlinear than their non-relativistic counterparts (relativistic equations of motion can always be rewritten in a form which coincides with that of the non-relativistic ones containing some effective energy and effective potential which are expressed via an initial potential). The study of chaotic relativistic dynamical systems could also be important when applied to real physical systems from relativistic cosmology, particle physics, physics of super-heavy atoms, etc.

Presently there are few works in which the chaotic properties of classical relativistic systems have been considered (Chernikov *et al* 1989, Luchinsky 1996, Kim Jung-Hoon and Lee Hai-Woong 1996, Drake *et al* 1996). In Matrasulov (1999) chaotic ionization of the relativistic hydrogen-like atom was considered on the basis of the resonance overlap criterion. Recently (Matrasulov 2001), chaotization of supercritical atoms (with nuclear charge $Z > 137$) has also been studied. This paper represents further progress in the study of chaos in relativistic atoms. Throughout the paper we use the relativistic system of units ($m_e = c = \hbar = 1$).

Consider a one-dimensional relativistic hydrogen-like atom with a nuclear charge $Z\alpha$ ($\alpha = \frac{1}{137}$) interacting with a monochromatic field with frequency ω. The Hamiltonian of this system has the form (in action-angle variables) (Matrasulov 1999)

$$H = \frac{n}{\sqrt{n^2 + Z^2\alpha^2}} + \epsilon x(n) \cos \omega t \tag{1}$$

where n is the action variable and $x(n)$ is the coordinate of the relativistic electron in action-angle variables. Following Casati *et al* (1988) we introduce the relativistic 'eccentric anomaly' ξ,

$$t = \left(\frac{n^2 + Z^2\alpha^2}{Z\alpha} \right)^{3/2} (\varepsilon\xi - \sin\xi)$$

$$x = (Z\alpha)^{-1} n \sqrt{n^2 + Z^2\alpha^2} (1 - \varepsilon^{-1} \cos\xi)$$

$$\theta = \varepsilon\xi - \sin\xi$$

where

$$\varepsilon = \frac{n}{\sqrt{n^2 + Z^2\alpha^2}}$$

is the full energy of the electron.

Then the Hamilton equations which have the forms

$$\frac{\mathrm{d}n}{\mathrm{d}t} = -\frac{\partial H}{\partial \theta} \qquad \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\partial H}{\partial n} \tag{2}$$

can be rewritten in terms of the new variables:

$$\frac{\mathrm{d}n}{\mathrm{d}\eta} = -\epsilon(n^2 + Z^2\alpha^2)\sin\xi\cos\omega t \tag{3}$$

$$\frac{\mathrm{d}\xi}{\mathrm{d}\eta} = -\frac{Z\alpha}{(n^2 + Z^2\alpha^2)^{3/2}} + \frac{2n^2 + Z^2\alpha^2}{\sqrt{n^2 + Z^2\alpha 2}}\epsilon\cos\omega t(1 - \varepsilon^{-1}\cos\xi) \tag{4}$$

where

$$\eta = \frac{(n^2 + Z^2\alpha^2)^{3/2}}{Z^2\alpha^2}(\xi + \pi). \tag{5}$$

Our purpose is to evaluate the change in action between two subsequent passages at the aphelion ($\xi = \pi$) integrating these equations approximately. Since we will perform this evaluation at first order in ϵ, we can neglect ϵ in the second- and third-order equations. Integrating the equations in the same way as in Casati *et al* (1988) we obtain

$$\omega t = \omega(n^2 + Z^2\alpha^2)^{3/2}(\varepsilon\xi - \sin\xi) + \phi. \tag{6}$$

The integration was started with $\eta = 0$ and $\xi = -\pi$. Therefore (Casati *et al* 1988)

$$\phi = \omega\left(t_0 + \pi\frac{n(n^2 + Z^2\alpha^2)}{Z^2\alpha^2}\right). \tag{7}$$

Inserting equation (6) into equation (3) and integrating over ξ we obtain

$$\begin{aligned}\Delta n &= -\epsilon\frac{(n^2 + Z^2\alpha^2)^{5/2}}{Z^2\alpha^2}\int_{-\pi}^{\pi}\cos\{\chi(\varepsilon\xi - \sin\xi) + \phi\}\sin\xi\,\mathrm{d}\xi \\ &= -\epsilon\frac{(n^2 + Z^2\alpha^2)^{5/2}}{Z^2\alpha^2}J'_\nu(\chi)\sin\phi\end{aligned} \tag{8}$$

where

$$\chi(n) = \frac{\omega(n^2 + Z^2\alpha^2)^{3/2}}{Z^2\alpha^2} \qquad \nu = \chi\varepsilon$$

$$J'_\nu(\chi) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sin(\xi(\varepsilon\xi - \sin\xi))\,\mathrm{d}\xi$$

is the derivative of the Anger function (Casati *et al* 1988, Abramowitz and Stegun 1964).

Following Casati *et al* (1988) we introduce a variable $N = (\varepsilon - 1)/\omega$, which describes, as in the non-relativistic case, the number of absorbed photons. Using equation (8) for the change of N, corresponding to the change of n we have

$$\begin{aligned}\Delta N &= \frac{\Delta n Z^2\alpha^2}{\omega(n^2 - Z^2\alpha^2)^{3/2}} \\ &= \frac{2\pi(n^2 + Z^2\alpha^2)\epsilon}{\omega}J'_\nu(\chi)\sin\phi = kA(\chi)\sin\phi\end{aligned} \tag{9}$$

where $k = 0.822\pi\epsilon(Z\alpha)^{4/3}\omega^{-5/3}$.

For the change of ϕ corresponding to the change of action n one has (from equation (7))

$$\Delta\phi = \frac{2\pi Z\alpha\omega x}{(1-x^2)^{3/2}} \tag{10}$$

where

$$x = \omega N - 1 = \frac{n}{\sqrt{n^2 + Z^2\alpha^2}}.$$

Following the standard procedure (see, e.g., Casati *et al* 1988, Kaulakis and Vilutis 1999) we will seek the generating function $G(\bar{N}, \phi)$ such that the map defined by

$$N = \frac{\partial G}{\partial \phi} \qquad \bar{\phi} = \frac{\partial G}{\partial \bar{N}}$$

coincides with equations (9) and (10) (in the first and zeroth order, respectively). The generating function obeying these conditions has the form:

$$G = \bar{N}\phi + 2\pi Z\alpha\omega(1-x^2)^{1/2} + kA(\bar{\chi})\cos\phi. \tag{11}$$

It generates the following relativistic Kepler map:

$$\bar{N} = N + kA(\bar{\chi})\sin\phi \tag{12}$$

$$\bar{\phi} = \phi - 2\pi Z\alpha\omega x(1-x^2)^{-3/2} + \frac{\partial}{\partial \bar{N}}[kA(\bar{\chi})]\cos\phi. \tag{13}$$

For $\chi \gg 1$, $A(\chi) \sim 1$ (Casati *et al* 1988); therefore, in this case it can be rewritten as

$$\bar{N} = N + k\sin\phi \tag{14}$$

$$\bar{\phi} = \phi - 2\pi Z\alpha\omega(1+\omega N)(-2\omega N - \omega^2 N^2)^{-3/2}. \tag{15}$$

As can be seen from these equations, in the non-relativistic limit this map coincides with the non-relativistic one. It is convenient to rewrite this map in terms of the dimensionless variable $E_0 = \omega N n_0^2/Z^2\alpha^2$:

$$\bar{E}_0 = E_0 + k_0\sin\phi \tag{16}$$

$$\bar{\phi} = \phi - 2\pi\omega_0(-2E_0)^{-3/2} f(E_0) \tag{17}$$

where

$$f(E_0) = \left(1 + \frac{Z^2\alpha^2}{n_0^2}\bar{E}_0\right)\left(1 + \frac{Z^2\alpha^2}{2n_0^2}\bar{E}_0\right)^{-3/2}. \tag{18}$$

For $f(E_0) = 1$ the non-relativistic limit is reached. Note that in the non-relativistic case in such a form the Kepler map does not depend on the charge of the atomic nuclei, i.e. we have the same phase-space portrait in the plane (E_0, ϕ) for hydrogen and uranium atoms. In figure 1 the phase-space portrait for the non-relativistic Kepler map in the variables E_0, ϕ is given. In figure 2 the corresponding relativistic Kepler map is plotted with the same (as in figure 2) values of ϵ_0 and ω_0 (for $n_0 = 6$, $Z = 92$). As can be seen from these figures, in the relativistic case the number of chaotic trajectories is more than in the corresponding non-relativistic case.

As is well known (Casati *et al* 1988, Jensen *et al* 1991), the Kepler map can be linearized about a scaled energy $n_0/2\omega_0$ and rewritten in the form of a standard map. Linearizing equation (15) in the same way as in the non-relativistic case we have the following (relativistic) standard map:

$$\bar{N} = N + k\sin\phi \tag{19}$$

$$\bar{\phi} = \phi + TN \tag{20}$$

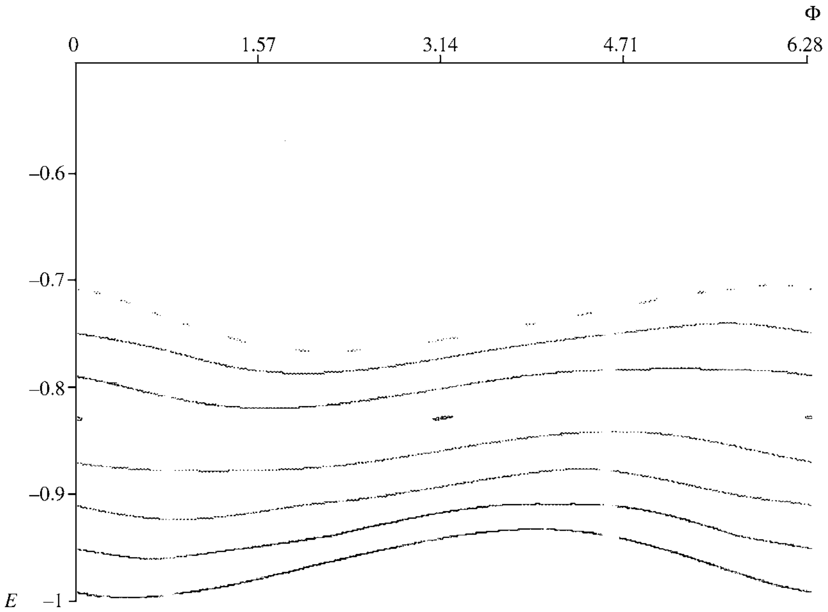

Figure 1. The phase-space portrait for the non-relativistic Kepler map in the variables E_0, ϕ ($\epsilon_0 = 0.03$, $\omega_0 = 3.5$).

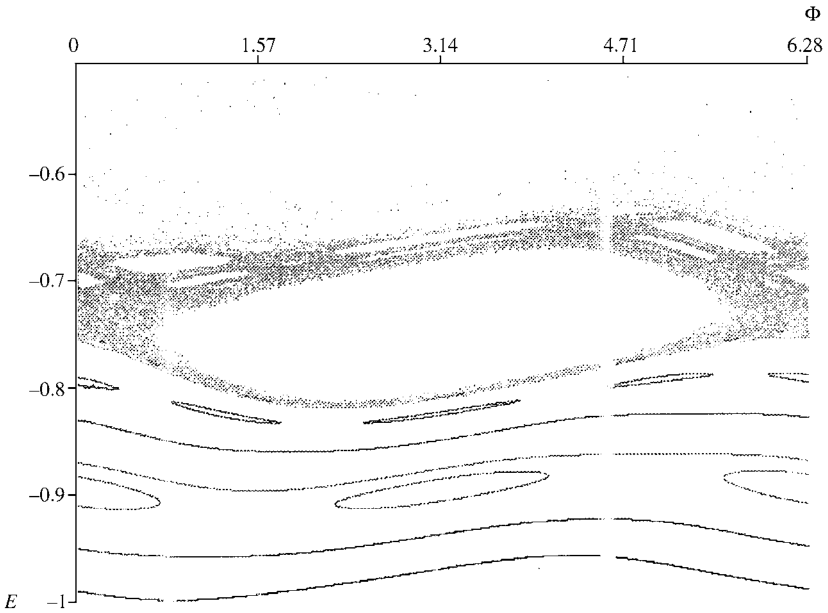

Figure 2. Relativistic Kepler map with the same (as in figure 1) values of ϵ_0 and ω_0 (for $n_0 = 6$, $Z = 92$).

where

$$T = T_{\mathrm{nonrel}}\left(1 - \frac{Z^2\alpha^2}{2n_0^2}\right)\left(1 - \frac{Z^2\alpha^2}{4n_0^2}\right)^{-3/2}\left[1 - \frac{Z^2\alpha^2}{2n_0^2}\frac{1 - Z^2\alpha^2/2n_0^2}{3\left(1 + Z^2\alpha^2/2n_0^2\right)\left(2 + Z^2\alpha^2/2n_0^2\right)}\right].$$

$T_{\mathrm{nonrel}} = 6\pi\omega^2 n_0^2$ is the corresponding coefficient for the non-relativistic case. The onset of stochasticity occurs when kT becomes larger than 1. From this condition one can estimate the critical value of the scaled field strength ($\epsilon_0 = \epsilon n^4$):

$$\epsilon_0 \approx \epsilon_{\mathrm{nonrel}}\left(1 - \frac{Z^2\alpha^2}{2n_0^2}\right)\left(1 - \frac{Z^2\alpha^2}{4n_0^2}\right)^{-3/2}\left[1 - \frac{Z^2\alpha^2}{2n_0^2}\frac{1 - Z^2\alpha^2/2n_0^2}{3\left(1 + Z^2\alpha^2/2n_0^2\right)\left(2 + Z^2\alpha^2/2n_0^2\right)}\right]$$

where $\epsilon_{\mathrm{nonrel}} = 1/49\epsilon_0\omega^{1/3}$ is the scaled field strength for the non-relativistic case.

Thus we have derived the relativistic generalization of the Kepler map describing the diffusion process of the periodically driven relativistic Kepler electron in energy space. As shown in figures 1 and 2 trajectories which are regular in the non-relativistic case can be chaotic in the relativistic case (with the same parameters as in the non-relativistic case). This can be explained by the fact that the relativistic Hamiltonian written in action-angle variables has a more complicated (nonlinear) form than the non-relativistic one. The respect in which this paper is an advance on previous ones (Matrasulov 1999, 2001) is that it raises the possibility of treating the chaotization process by plotting trajectories in phase space. In addition to this the linearized form of the Kepler map allows one to estimate the threshold field at which chaotization will occur. Another peculiarity of the obtained map is the fact that on the (E_0, ϕ) plane it depends on the charge of atomic nuclei; the corresponding non-relativistic Kepler map does not depend on the charge and gives the same phase-space portrait for hydrogen and uranium atoms. As is well known (Casati *et al* 1988, Jensen *et al* 1991, Leopold and Richards 1990), quantization of chaotic excitation and ionization of the hydrogen atom can be done with the classical Kepler map. Therefore, quantization of the relativistic Kepler map leads to a simplified quantal description of the chaotic ionization for the relativistic atom.

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